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ESTIMATION AND CONTROL FOR A SENSOR MOVING ALONG A ONE-DIMENSIONAL TRACK

by

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Abstract

We consider the problem of estimating a random process defined along a one-dimensional track using measurements from a sensor which traverses this track. The effects of sensor motion and motion blur on the estimation problem are considered, and in the particular case of a linear model for the random process and deterministic sensor motion, these effects are analyzed and discussed in detail. In this special case we also consider the problem of controlling the motion of the sensor in order to optimize some measure of the accuracy of our estimates along the track.

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I. Introduction

In this paper we consider the problem of recursive estimation of a random process defined along a one-dimensional track traversed by a moving sensor. Problems of this type arise in a variety of applications. For example, small variations in the gravitational field of the earth are often measured and mapped using data obtained from ships which travel along prescribed trajectories [1,2]. Another important context in which this kind of problem arises is in the remote sensing of atmospheric variables using instruments carried in a satellite [3-6], and a final related application is the processing of blurred images obtained from moving cameras [7].

In our work we have restricted attention to a special class of problems. Specifically, we focus on sensor motion along a one-dimensional track, on which the process to be estimated can be modeled as the output of a finite-dimensional shaping filter. While our general formulation allows for a nonlinear shaping filter, most of our attention will focus on the linear case.

These restrictions deserve some comment. Problems involving one-dimensional tracks occur in many applications. Data collected by a satellite along its trajectory is an obvious example, and gravity data from ships is also often collected along straight-line paths. These observations have led researchers [1-7] to use one-dimensional formulations for the various problems that they have considered. On the other hand, by restricting attention to one-dimensional tracks, we can expect to gain only some insights into the issues involved in mapping spatially-distributed random processes. The multidimensional problem clearly raises many questions which we have not considered and which must be in the future. Nevertheless, we feel that our study is a valuable step in gaining some understanding into problems of this type. In particular, the ideas and results that we have developed concerning the effect of sensor motion on the estimation problem are of some importance and, in fact, represent the major focus of our work.

The assumption that the random process to be estimated can be modeled as the output of a linear shaping filter is clearly an idealization. However, it is one that has found great use in practice [1-7]. For example, linear-gaussian models for the deviation of a gravitational field from some idealized reference have been developed using both physically-based models and statistical parameter identification techniques, and these models have proven to be of great value in practice [1,2]. The further assumption that the shaping filter model is finite dimensional is also an approximation. For example, physically-based models for the power spectral density of random gravity fluctuations are not rational [1,2], and, furthermore, except in certain special cases, the power spectral density along a track across a random field will not be rational even if the spectrum for the entire field is rational. Nevertheless, the assumption of finite-dimensionality is one that has met with success in applications, and we have chosen to use this assumption for this reason as well as for the reason of obtaining detailed solutions. The effects of sensor motion on these solutions are particularly clear, and this has facilitated our gaining an understanding of some of the issues that arise in processing data from moving sensors. Thus we feel that our study can be of value both in practice where approximations matching our assumptions can be made and in aiding our understanding into these types of problems. The removal of these assumptions remains for the future, but our work should provide a useful starting point.

A final point concerning the formulation and perspective adopted in this paper relates to the focus on <u>recursive</u> techniques. One of the largest problems to be faced in the analysis of spatially-distributed random data is that of efficient handling of the large amounts of data involved. Since model-based recursive estimation techniques have proven to be extremely efficient for processing time series data, it is natural to ask whether analogs of such techniques exist for spatial data. Our work is an initial look at adapting one-dimensional recursive techniques to spatial data processing. Clearly much work remains, especially in considering the substantial increase in problem complexity that will be encountered in more than one spatial dimension.

Thus the main goal of our work has been to gain some understanding into problems of mapping spatially-distributed random fields by considering the one-dimensional problem using the tools of recursive estimation theory. The major emphasis of this paper is on an examination of the effect of sensor motion on the estimation problem. In the next section we formulate the basic problem and indicate how sensor speed affects the measurements, while the specialization to the linear case is the topic addressed in Section III. The results of Section III are used in Section IV to formulate an optimal control problem for controlling sensor motion to achieve the best map possible. This formulation is very much in the spirit of the work in [8] on optimal search strategies. In Section V we extend the results of Section III to include the possibility of motion blur in the observations. Most of the detailed analysis through Section V is for the case of deterministic sensor motion. In Section VI we discuss the effects of random sensor motion, and the paper concludes with a discussion in Section VII of some of the issues we have raised and open problems that need to be examined.

II. Problem Formulation

Let s denote distance along the one-dimensional track, and let the (possibly vector-valued) spatial random process to be estimated be denoted by $\xi(s)$. Our basic assumption is that ξ can be modeled as the output of a spatial shaping filter, that is, a stochastic differential equation in s

$$dx(s) = f(x(s),s)ds + g(x(s),s)dw(s), \quad s \geq 0 \tag{2.1}$$

$$\xi(s) = h(x(s),s) \tag{2.2}$$

where $x(0)$ is a given random variable, independent of the Brownian motion process w which has covariance

$$E[w(s)w'(\sigma)] = \int_0^{\min(s,\sigma)} Q(\xi)d\xi \tag{2.3}$$

Note that if $\xi(s)$ has a rational power spectral density, we can always find a linear, space-invariant model of this type.

The spatial process is observed through a sensor that moves in the direction of increasing s with velocity $V(t)$. The velocity may be deterministic or random but is assumed to be positive for all t with probability 1. The equation of motion of the sensor then is

$$ds(t) = v(t)dt, \quad s(0)=0 \tag{2.4}$$

The value of the process ξ being observed at time t then is $\xi(s(t))$, and the measurements are modeled by*

$$dz_1(t) = r(\xi(s(t)),t)dt + d\beta_1(t) \tag{2.5}$$

where β_1 is a Brownian motion process with

* We include the subscript "1" here, as we will introduce a second set set of observations in Section VI.

$$E[\beta_1(t)\beta_1'(s)] = I \min(t,s) \tag{2.6}$$

We assume that $\{\beta_1(\tau_1) - \beta_1(\tau_2),\ \tau_1 \geq \tau_2 \geq t\}$ is independent of $\{s(\tau),\ V(\tau),\ w(s(\tau)),\ 0 \leq \tau \leq t\}$ and $x(0)$ and hence of $\{\tilde{x}(\tau),\ 0 \leq \tau \leq t\}$. Since $v(t)$ is positive, $s(t)$ is monotonically increasing and we can define $t(s)$ as the inverse of $s(t)$. We will assume that $w(s_1) - w(s_2)$, $s_1 > s_2 \geq s$, is independent of $\{s(\tau)\wedge s,\ \forall\tau \geq 0\} \cup \{v(t(s')),\ 0 \leq s' \leq s\}$.*

Since ξ is a memoryless function of x, we can combine equations (2.2) and (2.5) to obtain

$$dz_1(t) = c(\tilde{x}(t),\ s(t),t)dt + d\beta_1(t) \tag{2.7}$$

where

$$\tilde{x}(t) = x(s(t)) \tag{2.8}$$

$$c(\tilde{x}(t),s(t),t) = r[h(\tilde{x}(t),s(t)),t] \tag{2.9}$$

Our problem then is to estimate the spatial shaping filter state $x(s)$, which satisfies (2.1), (2.3), given the measurements z_1 specified by (2.6), (2.7), (2.8) and the sensor motion equation (2.4).

In order to solve this estimation problem, it is necessary to describe the evolution of $\tilde{x}(t)$. Note that if $x(s)$ were differentiable, we could write

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* A simpler but more restrictive condition would be that β_1 is independent of v, w, and $x(0)$ and that w is independent of v. The less restrictive condition given in the text is included since it allows for the possibility that the sensor velocity v might be chosen to depend upon past observations.

$$\frac{d}{dt}\,\tilde{x}(t) = v(t)\left.\frac{dx(s)}{ds}\right|_{s=s(t)} \tag{2.10}$$

However, in our case, we must utilize results on change of time scale for diffusion processes [9,10]. We present in Appendix 1 a statement of the required result without proof since this is a straightforward generalization of McKean's result [9]. An application of the result which requires $v(t) > 0$, $\forall t$, w.p.1, gives us

$$d\tilde{x}(t) = \tilde{f}(\tilde{x}(t),t)v(t)dt + \tilde{g}(\tilde{x}(t),t)v^{1/2}(t)d\eta(t) \tag{2.11}$$

where η is a Brownian notion process with

$$E[d\eta(t)^2] = \tilde{Q}(t)dt = Q(s(t))dt \tag{2.12}$$

and

$$\tilde{f}(\cdot,t) = f(\cdot,s(t)) \tag{2.13}$$

$$\tilde{g}(\cdot,t) = g(\cdot,s(t)) \tag{2.14}$$

The estimation of $\tilde{x}(t)$ is now a standard nonlinear filtering problem [11], which thus has all of the difficulties associated with that type of problem. A discussion of the general nonlinear case is given in [10]. For the remainder of this paper we will concentrate on the linear case.

III. Estimation of Linear Spatial Processes with Deterministic Sensor Motion

Suppose that we have a linear process model

$$dx(s) = A(s)x(s)ds + B(s)dw(s) \tag{3.1}$$

and linear observations

$$dz_1(t) = C(s(t),t)\tilde{x}(t) + d\beta_1(t) \tag{3.2}$$

In this case the evolution of $\tilde{x}(t)$ is given by

$$d\tilde{x}(t) = A(s(t))v(t)\tilde{x}(t)dt + B(s(t))v^{1/2}(t)d\eta(t) \tag{3.3}$$

Assuming that $v(t)$ is deterministic and that $x(0)$ is Gaussian with mean $\hat{x}(0)$ and variance $P(0)$, the conditional mean $\hat{\tilde{x}}(t)$ of $\tilde{x}(t)$ given $z_1(\tau)$, $\tau \leq t$ can be computed using the Kalman filter

$$d\hat{\tilde{x}}(t) = A(s(t))v(t)\hat{\tilde{x}}(t)dt + P(t)C'(s(t),t)[dz_1(t) - C(s(t),t)\hat{\tilde{x}}(t)dt] \tag{3.4}$$

The covariance $P(t)$ of the estimation error $(\tilde{x}(t)-\hat{\tilde{x}}(t))$ can be computed off-line from the Riccati equation

$$\dot{P}(t) = v(t)[A(s(t))P(t) + P(t)A'(s(t))] + v(t)B(s(t))Q(s(t))B'(s(t)) - P(t)C'(s(t),t)C(s(t),t)P(t) \tag{3.5}$$

Note that because of the assumption of deterministic sensor motion, the estimates $\hat{\tilde{x}}(t)$ can be directly transformed into estimates of the field $x(s)$. That is, $\hat{\tilde{x}}(t(s))$ is the optimal estimate of $x(s)$

given data up to the point s, or, equivalently, time t(s). The covariance of this estimate is obviously

$$M(s) = P(t(s)) \tag{3.6}$$

and, differentiating (3.6) we obtain

$$\frac{dM(s)}{ds} = A(s)M(s) + M(s)A'(s) + B(s)Q(s)B'(s) - \frac{M(s)C'(s,t(s))C(s,t(s))M(s)}{v(t(s))} \tag{3.7}$$

Examining (3.5) and (3.7) we can see how the speed of the sensor affects the performance of the estimator. The first two terms on the right-hand sides of (3.5) and (3.7) are the covariance propagation dynamics without measurements. Intuitively the matrix A(s) controls the "correlation distance" in the process x(s), while $A(s(t))v(t)$ determines the correlation time for $\tilde{x}(t)$. For example in the scalar, space-invariant (A=constant) case, $1/|A|$ is the correlation distance for x(s) and $1/|A|v$ is the correlation time for $\tilde{x}(t)$. Thus we have the physically correct feature that the faster we move, the faster the fluctuations we see in the observed process. Also, we would intuitively expect that the quality of the measurements would also decrease as the sensor velocity is increased. This feature can be deduced from (3.7), where we see that the term that tends to decrease M(s) to account for the observations is inversely proportional to v.

IV. Optimal Mapping via Sensor Motion Control

As we have seen, the motion of the sensor affects the quality of the observations being taken and hence the accuracy of the estimates. An interesting problem then is the control of sensor speed in order to optimize some measure of the quality of the spatial map that the observations produce. In this section we look at this problem and formulate an optimal control problem that captures the important features to be considered. We consider only the linear model - deterministic motion problem examined in the preceding section, and, for simplicity, we consider only the scalar case. Extension to the vector case is immediate using the matrix version of the minimum principle [14].

Suppose we define our measure of the quality of the spatial map on the interval $[0,s_0]$ by

$$\int_0^{s_0} q(s)M(s)ds \tag{4.1}$$

where $q(s)$ is a positive weighting function which we specify a priori. We also include a cost on sensor speed to reflect penalties for large velocities, and we assume that we have a fixed time interval $[0,T]$ in which we must traverse the spatial interval $[0,s_0]$. Then, transforming (4.1) to a time integral, we obtain the following optimal control problem. Given the dynamics

$$\frac{dP(t)}{dt} = 2A(s(t))v(t)P(t) + v(t)B^2(s(t))Q(s(t)) - C^2(t)P^2(t) \tag{4.2}$$

$$\frac{ds(t)}{dt} = v(t) \tag{4.3}$$

with given initial conditions

$$P(0) = P_0,\ s(0)=0 \tag{4.4}$$

determine the sensor velocity time history that minimizes

$$J = \int_0^T [q(s(t))v(t)P(t) + r(t)v^2(t)]dt \tag{4.5}$$

subject to

$$s(T) = s_0 \tag{4.6}$$

$$v(t) \geq \varepsilon \quad \forall t \tag{4.7}$$

Here, r(t) is a specified positive time function, and ε is an arbitrary but fixed positive number, included to insure the positivity of the velocity.

This optimal control problem can be solved by a direct application of the minimum principle [12,13]. We will consider this application with the inclusion of one more terminal condition:

$$P(T) = P_T \tag{4.8}$$

i.e., a type of "target" terminal estimation error. This terminal condition helps to simplify the two-point boundary value problem that must be solved to determine the optimal control. The free terminal condition problem can, of course, also be considered, but for our demonstration purposes we need only consider the simpler problem.

The Hamiltonian for our problem can now be written as

$$\begin{aligned} H = {} & D_0[q(s(t))P(t)v(t) + r(t)v^2(t)] \\ & + D_1(t)[2A(s(t))v(t)P(t) + v(t)B^2(s(t))Q(s(t)) \\ & - P^2(t)C^2(t)] \\ & + D_2(t)v(t) + \mu(t)[\varepsilon-v(t)] \end{aligned} \tag{4.9}$$

where

$$\mu(t) \begin{cases} \geq 0, & \varepsilon-v(t) = 0 \\ = 0, & \varepsilon-v(t) < 0 \end{cases} \tag{4.10}$$

(See [13].) The variables D_0, $D_1(t)$, $D_2(t)$ and $\mu(t)$ are costate variables. The optimal control problem can now be solved in principle by applying the minimum principle [12] to obtain the necessary conditions that characterize the optimal velocity $v^*(t)$ and the optimal estimation error covariance $p^*(t)$. These are given in Appendix 2. It is evidently impossible to obtain any algebraic simplification on the set of necessary conditions which, in practice, usually have to be solved numerically on a computer.

There are, however, special cases in which an explicit solution can be obtained, and we now present one such example. Assume the following constant conditions:

$$A = 0\ , \qquad B^2Q=1$$
$$r = 1/2, \qquad C^2 = 1/2 \tag{4.11}$$
$$q=1$$

In this case the process x(s) is a Wiener process, not a particularly realistic model, but it does allow us to obtain an explicit form for the solution. Note that the choice of $q(s)=1$ means that we give equal weight to the accuracy of all parts of our spatial map. Now, assume that the terminal conditions on P and s are so given that they can be met with more than *one velocity profile* $V(t)$, $0 \leq t \leq T$. Then, in the case in which $V(t) > \varepsilon \quad \forall\, t$, we can derive the following expression for $P^*(t)$:

$$\left(\frac{dP^*}{dt}\right)^2 = D_2^*(0)P^{*2} + P^{*3} + \frac{1}{4}P^{*4} + C \tag{4.12}$$

where

$$C = \left(\left.\frac{dP^*}{dt}\right|_{t=0}\right)^2 - (D_2^*(0)P^{*2}(0) + P^{*3}(0) + \frac{1}{4}P^{*4}(0)) \tag{4.13}$$

The derivation is presented in Appendix 3. By writing equation (4.12) as

$$ {\frac{dP^*}{dt}}^{2} = h^2 (P^* - \alpha)(P^* - \beta)(P^* - \gamma)(P^* - \delta), \qquad (4.14) $$

$$ h^2 = \frac{1}{4} $$

the solution is given by [15]

$$ P^*(t) = (\beta Y^2 - A\alpha)/(Y^2 - A) + (P^*(0) - \alpha) $$

where

$$ Y = \mathrm{sn}\{hMt,\ k\} \qquad (4.16) $$

$$ A = (\beta-\delta)/(\alpha-\delta) \qquad (4.17) $$

$$ k^2 = (\beta-\gamma)(\alpha-\delta)/(\alpha-\gamma)(\beta-\delta) \qquad (4.18) $$

$$ M^2 = (\beta-\delta)(\alpha-\gamma)/4 \qquad (4.19) $$

The function $\mathrm{sn}\{\cdot,\cdot\}$ is an elliptic function known as the sinus amplitudinus function [15] and it is tabulated in [16]. We have now obtained a closed form solution for $P^*(t)$, and this enables us to obtain the optimal velocity $v^*(t)$ from the Riccati equation, which is given in this case by

$$ \frac{dP^*(t)}{dt} = v^*(t) - \frac{1}{2} P^{*2}(t) \qquad (4.20) $$

V. The Inclusion of Motion Blur

We now suppose that because of its own dynamics, the sensor is not capable of making instantaneous, point measurements. Rather, the sensor output at time t involves a blurring of that part of the spatial process already swept

$$dz_1(t) = \left[\int_0^t H(t-\tau)\tilde{x}(\tau)d\tau\right]dt + d\beta_1(t) \tag{5.1}$$

where we have assumed, for simplicity, a time invariant blur model. Models of this type were considered in the discrete time case in [7].

Suppose that the matrix blurring function H is realizable as the impulse response of a finite dimensional linear system.

$$H(t-\tau) = Ce^{F(t-\tau)}G \tag{5.2}$$

Then we can write

$$dz_1(t) = Cy(t)dt + d\beta_1(t) \tag{5.3}$$

$$dy(t) = Fy(t)dt + G\tilde{x}(t)dt \tag{5.4}$$

We now have an estimation problem with an augmented state, consisting of $\tilde{x}$ and y, and the optimal filtering equations are

$$d\begin{bmatrix}\hat{\hat{x}}(t)\\ \hat{y}(t)\end{bmatrix} = \begin{bmatrix}A(s(t))v(t) & 0\\ G & F\end{bmatrix}\begin{bmatrix}\hat{\hat{x}}(t)\\ \hat{y}(t)\end{bmatrix}dt$$

$$+ P(t)[0,C']\{dz_1(t) - C\hat{y}(t)dt\} \tag{5.5}$$

where P(t), the error covariance for the augmented state estimation error can be computed from

$$\dot{P}(t) = \begin{bmatrix}A(s(t))v(t) & 0\\ G & F\end{bmatrix}P(t) + P(t)\begin{bmatrix}A'(s(t))v(t) & G'\\ 0 & F'\end{bmatrix}$$

$$+\begin{bmatrix}v(t)B(s(t))Q(s(t))B'(s(t)) & 0\\ 0 & 0\end{bmatrix} - P(t)\begin{bmatrix}0 & 0\\ 0 & C'C\end{bmatrix}P(t) \tag{5.6}$$

VI. The Effect of Imperfectly Known Sensor Motion

The analysis in the last few sections has been aided by the assumption that the trajectory of the sensor was known or perfectly controllable. In this section we indicate some of the complications that arise if this is not the case. We assume that the spatial process is modeled as in (3.3), which is repeated here for convenience:

$$d\tilde{x}(t) = A(s(t))v(t)\tilde{x}(t)dt + B(s(t))v^{1/2}(t)d\eta(t) \tag{6.1}$$

and we assume that the motion of the sensor can be described by

$$ds(t) = v(t)dt \tag{6.2}$$

$$dv(t) = u(t)dt + k(v(t),t)d\xi(t) \tag{6.3}$$

Here $u(t)$ represents the known part of the sensor's acceleration, while the other term models the unknown random perturbations in the velocity. Here ξ is a standard Brownian motion process. Note that for our formulation, possible choices of k are restricted to those for which $v(t) > 0 \ \forall t$ with probability 1. For example, the bilinear model

$$k(v) = -\alpha v \tag{6.4}$$

with the assumption $u \geq 0$, $v(0) > 0$ satisfies the positivity condition. In general, we must have k dependent upon v to satisfy the constraint, and this rules out a linear model. Of course if u is large compared with the disturbance, we may be able to use the linear model in practice.

Given the model (6.1)-(6.3), we assume that we observe

$$dz_1(t) = c(t)\tilde{x}(t)dt + d\beta_1(t) \tag{6.5}$$

$$dz_2(t) = v(t)dt + d\beta_2(t) \tag{6.6}$$

$$dz_3(t) = s(t)dt + d\beta_3(t) \tag{6.7}$$

where β_2, and β_3 are independent Wiener process both independent of β_1. Our goal is to obtain a spatial map of the process $x(s)$ given the observations $Z^t = \{z_1(\tau), z_2(\tau), z_3(\tau), \tau \leq t\}$ unfortunately, two types of problems occur. First of all, the optimal estimation of $\tilde{x}$, s, and v is a nonlinear filtering problem, and this is the case even if A, B, and Q do not depend on s and we assume a linear model in (6.3).

The problem is the product terms in (6.1), since v is now random. Note also that all of the observations contain information about all of the states. For example, the observation z_1 does yield information concerning the velocity v (and hence the position s). In fact, it is precisely this information that is used in map-matching navigation systems [1,20] in which position and velocity are deduced by correlating an a priori map of the process $x(s)$ with the observed process $z_1(t)$.

The second problem centers around the issue of mapping itself. Recall that

$$\hat{\hat{x}}(t) = E[x(s(t))\,|\,Z^t] \tag{6.8}$$

When $s(t)$ was known perfectly, we could associate this estimate with a specific spatial point. That is,

$$\hat{x}(s) = E[x(s)\,|\,Z^{t(s)}] = \hat{\hat{x}}(t(s)) \tag{6.9}$$

However, when s itself is unknown and must be estimated, we do not have such a simple relationship, and, in fact, we can not exactly associate $\hat{\hat{x}}(t)$ with the estimate of $x(s)$ at any specific point.

To overcome this difficulty, one might consider estimating $x(\tilde{s}(t))$ where $\tilde{s}(t)$ is measurable with respect to z^t (and hence is known when we know the measurements). Such an approach leads to some extremely complex technical problems. For example, one might consider trying to estimate $x(\hat{s}(t))$, where

$$\hat{s}(t) = E[s(t) \mid Z^t] \tag{6.10}$$

However, we cannot obtain a differential equation for $x(\hat{s}(t))$ as we did for $x(s(t))$. The problem is that in the latter case we changed the time scale of a diffusion process with an increasing process $s(t)$. In the case of $x(\hat{s}(t))$ we want to change the time scale of a diffusion process using another diffusion process (which, of course, need not even be increasing). We refer the reader to [10,17] for further discussion of these technical problems and several other approaches.

In the remainder of this section we describe one suboptimal estimation scheme that arises naturally from our formulation and from the analysis of the preceding sections. This scheme decouples the sensor location and field estimation problems. Suppose we compute the estimates of $v(t)$ and $s(t)$ using only the observations z_2 and z_3. If we make the assumption that (6.3) is linear ($k(v(t),t)=g$), these estimates are calculated by a Kalman filter

$$\begin{bmatrix} d\,\bar{s}(t) \\ d\,\bar{v}(t) \end{bmatrix} = \begin{bmatrix} \bar{v}(t) \\ u(t) \end{bmatrix} dt + K(t) \begin{bmatrix} dz_3(t) - \bar{s}(t)dt \\ dz_2(t) - \bar{v}(t)dt \end{bmatrix} \tag{6.11}$$

where, assuming that β_2 and β_3 are unit strength and independent, $K(t)$ satisfies the Riccati equation

$$\dot{K}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} K(t) + K(t) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & g^2 \end{bmatrix} - K^2(t) \tag{6.12}$$

Having the estimates $\bar{s}(t)$ and $\bar{v}(t)$, we now devise an estimate for $\tilde{x}(t)$ assuming that these values of $\bar{s}(t)$ and $\bar{v}(t)$ are, in fact, the true values. That is, we implement the Kalman filter of Section III with v and s replaced by $\bar{v}$ and $\bar{s}$. This yields the filter equations

$$d\bar{\tilde{x}}(t) = A(\bar{s}(t))\bar{v}(t)\bar{\tilde{x}}(t)dt + \bar{P}(t)C'(\bar{s}(t),t)[dz_1(t) - C(\bar{s}(t),t)\bar{\tilde{x}}(t)dt] \tag{6.13}$$

$$\frac{d\bar{P}(t)}{dt} = \bar{v}(t)[A(\bar{s}(t))\bar{P}(t) + \bar{P}(t)A'(\bar{s}(t))] + \bar{v}(t)B(\bar{s}(t))Q(\bar{s}(t))B'(\bar{s}(t)) - \bar{P}(t)C'(\bar{s}(t),t)C(\bar{s}(t),t)\bar{P}(t) \tag{6.14}$$

Note that the Riccati equation (6.14) must be solved on-line, as the quality of the measurements -- as dictated by sensor speed -- is estimated on-line. We alsc associate the estimate $\bar{\tilde{x}}(t)$ with the point $\bar{s}(t)$ on our spatial map. In theory, there is no guarantee that $\bar{s}(t)$ is monotonically increasing but in practice it is very likely to be so because position estimates can often be made very accurately. An evaluation of the performance of this estimator and the development of alternative schemes including those that attempt to extract velocity and position information from the observations z_1 remain for the future.

VII. Conclusions

In this paper we have formulated and studied the problem of estimating a one-dimensional time invariant spatial random process given observations from a moving point sensor. Our formulation has allowed us to study the effects of sensor motion on the quality of the observations and on the estimation problem itself. This has led us to consider the problem of optimally controlling the velocity of the sensor and to study the effects of uncertainties in our knowledge of sensor location and speed. In addition, we have shown how our formulation can be extended to allow for the effects of sensor blurring.

As mentioned in the introduction, our purpose here has been to expose some of the key issues involved and to provide a foundation for further, more advanced studies. Several extensions and related problems directly come out of the questions we have studied. An obvious area for further work is in the study of the nature and structure of the optimal velocity control problem discussed in Section IV. In addition, one might also wish to consider the problem in which the control variable is sensor acceleration. In this case v is a state variable, and, because of (4.7), we have a state-constrained optimal control problem. Also, in the nonlinear case or the uncertain motion problem of Section VI, the optimal velocity or acceleration problem becomes one of on-line stochastic control. The structure of such controllers should be investigated, as should the performance of the estimator suggested in Section VI either by analysis or by simulations.

Another variation that brings us closer to a realistic formulation for many problems, is to replace the filtered covariance $P(t)$ in the mapping criterion (4.2) with the smoothed covariance, i.e., we utilize the entire measurement history $z_1(\tau)$, $\tau\varepsilon[0,T]$ to obtain an accurate spatial map over the region $[0,s_0]$. In this case we lose the causal structure (the smoothed error covariance depends upon the entire velocity history), and the study of the nature of optimal sensor trajectories in this case is an interesting problem. Also we can consider extending our analysis by allowing the sensor to reverse direction. The deterministic analysis of Section III can clearly be extended in this case, although the optimal estimator immediately becomes a smoother once the sensor goes into reverse. In the case of random sensor motion, even the time of the reversal of direction is unknown, and hence we do not even know when to start smoothing. The study of this is open. Intuitively, if we use a criterion based on smoothed error covariances, one would expect that any performance achievable by a trajectory with reverse motion can also be achieved by a monotone trajectory. The study of problems such as these remains open.

In the introduction we mentioned that the sensor motion control problem is similar in spirit to the results in [8] on optimal search problems. In the formulations in [8] one is interested in determining strategies for searching a region for some object, given a specification of the probability of the detection of the object in a subset of the region as a function of the amount of energy put into searching that subset. In our formulation the velocity-estimation error covariance relationship plays the role of the search energy-probability of detection specification. Given this observation an interesting problem is the following: suppose we modify the description of x(s) as in (3.1) by allowing for one or more jumps in the value of x(s) at unknown locations; determine the optimal search procedure -- i.e. velocity profile -- to locate these jumps. Here again one might imagine on-line procedures, where we may choose to reverse direction to look at a given region more carefully once we've satisfied ourselves that no jumps are present outside that region. In this case some of the techniques for the detection of failures and other abrupt changes may be of value [19].

As mentioned in Section VI, the problem of estimation when sensor motion is uncertain represents a difficult challenge. Not only should the suboptimal estimator discussed be studied, but there is certainly a need for the development of other estimation systems. Of particular importance is the problem of estimating s(t) and v(t) given the sensor measurements $z_1(t)$. As we discussed earlier, this is a problem of potentially great practical significance for map-matching navigation systems. Another important possibility is to allow the spatial process to directly affect sensor motion [10,18]. This might arise, for example if the spatial process were a force field (such as a gravitational field) and our only observations were of the motion of the "sensor" (i.e., only z_2 and z_3 of Section VI). In this case it is the field x(s) which is observed only indirectly through its influence on v(t) and s(t).

Finally, there are the extensions of these ideas to processes that vary in several spatial dimensions and possibly in time. Problems such as estimation given data along one or more tracks each of which can contain changes of direction, curves, crossings, etc., are of importance in applications such as gravity mapping and meteorological analysis. In these as in many multidimensional problems two of the central difficulties are the lack of an efficient procedure for assimilating all of the data and the absence of a method for devising efficient strategies for deciding what data should be gathered. The results in this paper are aimed at the simplest of problems of these types and thus merely form an initial step. In order for the more general cases to be considered, a substantial effort is needed in obtaining useful multidimensional probabilistic models and formulations.

APPENDIX 1

Random Change of Time for a Brownian Motion

Theorem: Let (Ω,F,P) be a probability space, $\{F_s, s \geq 0\}$ an increasing family of sub-σ-algebras of F, and let $w(s)$ be a Brownian motion with respect to F_s with covariance given by (2.3). Let $v(t)$, $t \geq 0$ be a positive random process, and let $s(t)$ be given by (2.4). Let $t(s)$ denote the inverse of $s(t)$. Furthermore, assume that the increments $w(s_1)-w(s_2)$, $s_1 > s_2 \geq s$, are independent of $\{s(\tau)\wedge s, \tau \geq 0\}$ and $\{v(t(s')), 0 \leq s' \leq s\}$. Define the increasing family of σ-algebras

$$G_s = F_s \bigvee \sigma\{s(\tau)\wedge s, \quad \tau > 0\} \bigvee \sigma\{v(t(s')), 0 \leq s' \leq s\} \tag{A.1}$$

Then for each t, $s(t)$ is a stopping time with respect to G_s, and on the family $\{\tilde{G}_t\}_{t\geq 0}$, where

$$\tilde{G}_t = G_{s(t)} \tag{A.2}$$

the process

$$\tilde{w}(t) = w(s(t)) \tag{A.3}$$

is a martingale with respect to time t and is given by

$$d\tilde{w}(t) = v^{1/2}(t)d\eta(t) \tag{A.4}$$

where $\{\eta_t, \tilde{G}_t\}$ is a Wiener process with respect to time t with $E\{d\eta^2(t)\} = \tilde{Q}(t)dt = Q(s(t))dt$.

For a proof, see [9,10].

APPENDIX 2

Necessary Conditions for Optimality in the Sensor Control Problem

In order that $v^*(t)$ be optimal, it is necessary that the following conditions be satisfied (* denotes optimal):

(a) $$\frac{dP^*(t)}{dt} = 2A(s^*(t))v^*(t)P^*(t) + v^*(t)B^2(s^*(t))Q(s^*(t)) - P^{*2}(t)C^2(t) \tag{A.5}$$

$$\frac{ds^*(t)}{dt} = v^*(t) \tag{A.6}$$

$$D_0^* \geq 0 \tag{A.7}$$

$$\frac{dD_1^*(t)}{dt} = - \left.\frac{\partial H}{\partial P}\right|_* = - D_0^* q(s^*(t))\,v^*(t) - 2D_1^*(t)\,A(s^*(t))\,v^*(t) + 2P^*(t)\,C^2(t)D_1^*(t) \tag{A.8}$$

$$\frac{dD_2^*(t)}{dt} = - \left.\frac{\partial H}{\partial s}\right|_*$$

$$= - D_0^* P^*(t) v^*(t) \frac{\partial q}{\partial s} (s^*(t))$$

$$- 2D_1^*(t) v^*(t) P^*(t) \frac{\partial A}{\partial s} (s^*(t)) \qquad \text{(A.9)}$$

$$- 2D_1^*(t) v^*(t) B(S^*(t)) \frac{\partial B}{\partial s} (s^*(t)) Q(s^*(t))$$

$$- D_1^*(t) v^*(t) B^2(s^*(t)) \frac{\partial Q}{\partial s} (s^*(t))$$

$$P^*(0) = \sigma_0 \ , \quad P^*(T) = P_T \qquad \text{(A.10)}$$

$$s^*(0) = 0, \qquad s^*(T) = s_0 \qquad \text{(A.11)}$$

(b) Minimization of H with respect to v:

$$\left.\frac{\partial H}{\partial v}\right|_* = 0 = D_0^* q(s^*(t)) P^*(t) + 2D_0^* r(t) v^*(t)$$

$$+ 2D_1^*(t) A(s^*(t)) P^*(t)$$

$$+ D_1^*(t) B^2(s^*(t)) Q(s^*(t)) \qquad \text{(A.12)}$$

$$+ D_2^*(t) - \mu^*(t)$$

Since

$$\frac{\partial^2 H}{\partial v^2} = 2r(t) \ D_0^* \geq 0 \qquad \text{(A.13)}$$

we conclude that v^* obtained from equation (A.12) must necessarily minimize H. Equation (A.12) gives us only one solution for v^* so this must necessarily be a global minimum.

APPENDIX 3

Derivation of Equation (4.12)

For the special case given by equation (4.11), the necessary conditions become:

(a)
$$\frac{dP^*(t)}{dt} = v^*(t) - \frac{1}{2} P^{*2}(t) \tag{A.14}$$

$$\frac{ds^*(t)}{dt} = v^*(t) \tag{A.15}$$

$$D_0^* \geq 0 \tag{A.16}$$

$$\frac{dD_1^*(t)}{dt} = - D_0^* v^*(t) + P^*(t) D_1^*(t) \tag{A.17}$$

$$\frac{dD_2^*(t)}{dt} = 0 \tag{A.18}$$

(b) Minimization of H with respect to v:

$$\left.\frac{\partial H}{\partial v}\right|_* = 0 = D_0^* P^*(t) + D_0^* v^*(t) + D_1^*(t) + D_2^*(t) - \mu^*(t) \tag{A.19}$$

Since we assume that the terminal conditions on P and s are so given that they can be met with more than one velocity profile v(t), $0 \leq t \leq T$, we can set

$$D_0^* = 1 \tag{A.20}$$

In the case when $v(t) > \varepsilon$, we set

$$\mu^*(t) = 0 \tag{A.21}$$

Equation (A.19) then gives

$$v^*(t) = - P^*(t) - D_1^*(t) - D_2^*(t) \tag{A.22}$$

Differentiate this and substitute from (A.14), (A.17) and (A.18) to obtain

$$\frac{dv^*(t)}{dt} = \frac{1}{2} P^{*2}(t) - P^*(t)D_1^*(t) \tag{A.23}$$

Using $D_1^*(t)$ from (A.22) and noting that

$$D_2^*(t) = D_2^*(0) \tag{A.24}$$

we find that

$$\frac{dv^*(t)}{dt} = \frac{3}{2} P^{*2}(t) + P^*(t)(v^*(t) + D_2^*(0)) \tag{A.25}$$

Next use $v^*(t)$ from (A.14) to obtain

$$\frac{dv^*(t)}{dt} = P^*(t)\left(\frac{dP^*(t)}{dt} + D_2^*(0)\right) + \frac{3}{2} P^{*2}(t) + \frac{1}{2} P^{*3}(t) \tag{A.26}$$

Finally, differentiate (A.14) and substitute from (A.26) to get

$$\frac{d^2P^*(t)}{dt^2} = D_2^*(0)P^*(t) + \frac{3}{2}\; P^{*2}(t) + \frac{1}{2}\, P^{*3}(t) \tag{A.27}$$

This is a differential equation in $P^*(t)$. Multiplying the left side by $2\,\frac{dP^*}{dt}\,dt$ and the right side by $2dP^*$ gives

$$\frac{d}{dt}\left(\frac{dP^*}{dt}\right)^2 dt = 2(D_2^*(0)P^* + \frac{3}{2}\,P^{*2} + \frac{1}{2}\,P^{*3})dP^* \tag{A.28}$$

An integration gives equation (4.12).

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